CARTESIAN CLOSED 2-CATEGORIES AND PERMUTATION EQUIVALENCE IN HIGHER-ORDER REWRITING

TOM HIRSCHOWITZ

Abstract. We propose a semantics for permutation equivalence in higher-order rewriting. This semantics takes place in cartesian closed 2-categories, and is proved sound and complete.

1. Introduction

It is known since the end of the 80's that 2-categories with finite products provide a semantics for term rewriting [3]. *Higher-order rewriting* [10, 17, 14, 15] is a framework for specifying rewrite systems on terms with variable binding. Many results from standard term rewriting have been generalised to higher-order rewriting, notably normalisation or confluence results. An important tool for confluence results is the notion of *permutation equivalence*, which was generalised to the higher-order case by Bruggink [1]. He defines a calculus of *proof terms* for specifying reductions in a higher-order rewrite system.

We here propose a categorical semantics for a variant of this calculus, in terms of *cartesian closed 2-categories*. We first define *cartesian closed 2-signatures*, which generalise higher-order rewrite systems, and organise them into a category Sig. We then construct an adjunction

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{H}"] \arrow[r, phantom, "\perp"] & \mathsf{2CCCat}, \arrow[l, bend left, "\mathcal{W}"] \end{tikzcd} \tag{1.1}$$

where $\mathsf{2CCCat}$ is the category of small cartesian closed 2-categories. From a given higher-order rewrite system S, the functor $\mathcal{H}$ constructs a cartesian closed 2-category, whose 2-cells are Bruggink's proof terms modulo permutation equivalence, which we prove is the free cartesian closed 2-category generated by S.

We review a number of examples and non-examples, and sketch an extension to deal with the latter.

Related work. Our cartesian closed 2-signatures are a 2-dimensional refinement of *cartesian closed sketches* [16, 4, 9]. Bruggink's calculus of permutation equivalence is close in spirit to Hilken's 2-categorical semantics of the simply-typed λ-calculus [7], but technically different and generalised to arbitrary higher-order rewrite systems. Capriotti [2] proposes a semantics of so-called *flat* permutation equivalence in sesquicategories. More related work is discussed in Section 4.2.

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2. Cartesian closed signatures and categories

We start by recalling the well-known, or at least folklore, adjunction between what we here call *(cartesian closed) 1-signatures* and cartesian closed categories.

For any set X, define *types* over X by the grammar:

$$A, B, \ldots \in \mathcal{L}_0(X) \quad ::= \quad x \mid 1 \mid A \times B \mid B^A,$$

with $x \in X$.

Proposition 1. *$\mathcal{L}_0$ defines a monad on* Set.

Let the set of *sequents* over a set X be $\mathcal{S}_0(X) = \mathcal{L}_0(X)^* \times \mathcal{L}_0(X)$, i.e., sequents are pairs of a list of types and a type. The assignment $X \mapsto \mathcal{S}_0(X)$ extends to an endofunctor on Set.

Definition 1. *A* 1-signature *consists of a set X_0 of* sorts*, and an $\mathcal{S}_0(X_0)$-indexed set X_1 of* operations*, or equivalently a map $X_1 \to \mathcal{S}_0(X_0)$.*

A *morphism of 1-signatures* $(X_0, X_1) \to (Y_0, Y_1)$ is a pair (f_0, f_1) where $f_i \colon X_i \to Y_i$ such that

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ \mathcal{S}_0(X_0) \arrow[r, "\mathcal{S}_0(f_0)"'] & \mathcal{S}_0(Y_0) \end{tikzcd}$$

commutes. Morphisms compose in the obvious way, and we have:

Proposition 2. *Composition of morphisms is associative and unital, and hence 1-signatures and their morphisms form a category* Sig_1.

There is a well-known adjunction

$$\begin{tikzcd} \mathsf{Sig}_1 \arrow[r, bend left, "\mathcal{H}_1"] \arrow[r, phantom, "\perp"] & \mathsf{CCCat} \arrow[l, bend left, "\mathcal{W}_1"] \end{tikzcd}$$

between 1-signatures and the category CCCat of small cartesian closed categories (with chosen structure) and (strict) cartesian closed functors.

The functor $\mathcal{W}_1$ sends a cartesian closed category $\mathcal{C}$ to the signature with sorts $\mathcal{C}_0$, its set of objects, and with operations $A_1, \ldots, A_n \to A$ the set $\mathcal{C}([\![A_1 \times \ldots \times A_n]\!], [\![A]\!])$, where $[\![-]\!]$ denotes the function $\mathcal{L}_0(\mathcal{C}_0) \to \mathcal{C}_0$ defined by induction:

$$\begin{array}{rcll} [\![c]\!] & = & c & c \in \mathcal{C}_0 \\ [\![1]\!] & = & 1 & \\ [\![A \times B]\!] & = & [\![A]\!] \times [\![B]\!] & \\ [\![B^A]\!] & = & [\![B]\!]^{[\![A]\!]}. & \end{array} \tag{2.1}$$

Conversely, given a 1-signature X, consider the simply-typed λ-calculus with base types in X_0 and constants in X_1. Terms modulo $\beta\eta$ form a category $\mathcal{H}_1(X)$ with objects all types over X_0 and morphisms $A \to B$ all terms of type B with one free variable of type A.

A less often formulated observation, which is useful to us, is that the adjunction $\mathcal{H}_1 \dashv \mathcal{W}_1$ decomposes into two adjunctions

$$\begin{tikzcd} \mathsf{Sig}_1 \arrow[r, bend left, "\mathcal{L}_1"] \arrow[r, phantom, "\perp"] & \mathcal{L}_1\text{-Alg} \arrow[l, bend left, "\mathcal{U}_1"] \arrow[r, bend left, "\mathcal{F}_1"] \arrow[r, phantom, "\perp"] & \mathsf{CCCat}, \arrow[l, bend left, "\mathcal{V}_1"] \end{tikzcd}$$

where $\mathcal{L}_1$-Alg is the category of algebras for the monad $\mathcal{L}_1$ defined as follows (and $\mathcal{L}_1$ is shorthand for the functor $X \mapsto (\mathcal{L}_1(X), \mu)$).

For any 1-signature X, let $\mathcal{L}_1(X)$ denote the 1-signature with

- as sorts the set X_0, and
- as operations $\Gamma \vdash A$ the λ-terms $\Gamma \vdash M \colon A$, modulo $\beta\eta$.

$\mathcal{L}_1$ extends to an endofunctor on Sig_1, whose action on morphisms of 1-signatures $X \xrightarrow{f} Y$ substitutes constants $c \in X_1$ with $f_1(c)$. We obtain

Proposition 3. *$\mathcal{L}_1$ is a monad on Sig_1.*

The functor $\mathcal{V}_1$ sends any cartesian closed category $\mathcal{C}$ to the $\mathcal{L}_1$-algebras $(\mathcal{C}_0, \mathcal{C}_1)$ defined as follows. First, $\mathcal{C}_0$ is the set of objects of $\mathcal{C}$. It has a canonical $\mathcal{L}_0$-algebra structure, say $h_0 \colon \mathcal{L}_0(\mathcal{C}_0) \to \mathcal{C}_0$, obtained by interpreting type constructors in $\mathcal{C}$ as in (2.1). Extending this to contexts G by $h_0(G) = \prod_i h_0(G_i)$, let the operations in $\mathcal{C}_1(G, A)$ be the 1-cells in $\mathcal{C}(h_0(G), h_0(A))$. Beware: the domain and codomain of such an operation are really G and A, not $h_0(G)$ and $h_0(A)$. Similarly, interpreting the λ-calculus in $\mathcal{C}$, the 1-signature $(\mathcal{C}_0, \mathcal{C}_1)$ has a canonical $\mathcal{L}_1$-algebra structure, say $h_1 \colon \mathcal{L}_1(\mathcal{C}_0, \mathcal{C}_1) \to (\mathcal{C}_0, \mathcal{C}_1)$:

$$\begin{array}{rcl} h_1(G \vdash x_i \colon G_i) & = & \pi_i \\ h_1(G \vdash () \colon 1) & = & ! \\ h_1(G \vdash c(M_1, \ldots, M_n)) & = & c \circ (h_1(M_1), \ldots, h_1(M_n)) \\ h_1(G \vdash \lambda x \colon A.M \colon B^A) & = & \varphi(h_1(G, x \colon A \vdash M \colon B)) \\ h_1(G \vdash MN \colon B) & = & ev \circ (h_1(M), h_1(N)) \\ h_1(G \vdash (M, N) \colon A \times B) & = & (h_1(M), h_1(N)) \\ h_1(G \vdash \pi M \colon A) & = & \pi \circ M \\ h_1(G \vdash \pi' M \colon A) & = & \pi' \circ M, \end{array}$$

where ! is the unique morphism $h_0(G) \to 1$, φ is the bijection $\mathcal{C}(h_0(G, A), h_0(B)) \cong \mathcal{C}(h_0(G), h_0(B^A))$, and ev is the structure morphism $h_0(B^A \times A) \to h_0(B)$.

$\mathcal{L}_1$-algebras are much like cartesian closed categories whose objects are freely generated by their set of sorts. A perhaps useful analogy here is with multicategories $\mathcal{M}$, seen as being close to monoidal categories whose objects are freely generated by those of $\mathcal{M}$ by tensor and unit. Here, the functor $\mathcal{F}_1$ sends any $\mathcal{L}_1$-algebra (X, h) to the cartesian closed category with

- objects the types over X_0, i.e., $\mathcal{L}_0(X_0)$,
- morphisms $A \to B$ the set of operations in $X_1(A, B)$.

This canonically forms a cartesian closed category, with structure induced by the $\mathcal{L}_1$-algebra structure. We define it in more detail in dimension 2 in Section 7.2.

3. Cartesian closed 2-signatures

Given a 1-signature X, let $X_{||}$ denote the set of pairs of parallel operations, i.e., pairs of operations M, N above the same sequent. Otherwise said, $X_{||}$ is the pullback

$$\begin{tikzcd} X_{||} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d] \\ X_1 \arrow[r] & \mathcal{S}_0(X_0). \end{tikzcd}$$

Any morphism $f\colon X \to Y$ of 1-signatures yields a function $f_{||}\colon X_{||} \to Y_{||}$, via the dashed arrow (obtained by universal property of pullback) in

$$\begin{tikzcd}
X_{||} \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[ddrr, phantom, "\lrcorner", very near start] & & X_1 \arrow[dd] \arrow[dr, "f_1"] & \\
& Y_{||} \arrow[rr, crossing over] \arrow[ddrr, phantom, "\lrcorner", very near start] & & Y_1 \arrow[dd] \\
X_1 \arrow[rr] \arrow[dr, "f_1"'] & & \mathcal{S}_0(X_0) \arrow[dr, "\mathcal{S}_0(f_0)"] & \\
& Y_1 \arrow[rr] \arrow[from=uu, crossing over] & & \mathcal{S}_0(Y_0).
\end{tikzcd}$$

Definition 2. *A* 2-signature *consists of a 1-signature X, plus a set X_2 of* reduction rules *with a function $X_2 \to \mathcal{L}_1(X)_{||}$.*

A *morphism of 2-signatures* $(X, X_2) \to (Y, Y_2)$ is a pair (f, f_2) where $f\colon X \to Y$ is a morphism of 1-signatures and $f_2\colon X_2 \to Y_2$ makes the diagram

$$\begin{tikzcd}
X_2 \arrow[r, "f_2"] \arrow[d] & Y_2 \arrow[d] \\
\mathcal{L}_1(X)_{||} \arrow[r, "\mathcal{L}_1(f_1)_{||}"'] & \mathcal{L}_1(Y)_{||}
\end{tikzcd}$$

commute. We obtain:

Proposition 4. *Composition of morphisms is associative and unital, and hence 2-signatures and their morphisms form a category* Sig.

4. Examples

4.1. **Higher-order rewrite systems.** The prime example of a 2-signature is that for the pure λ-calculus: it has a sort t and operations

$$a\colon t \times t \to t \qquad\qquad \ell\colon t^t \to t,$$

with a reduction rule β above the pair

$$x\colon t^t, y\colon t \vdash a(\ell(x), y), x(y)\colon t$$

in $\mathcal{L}_1(\{t\}, \{\ell, a\})_{||}$. Categorically, this will yield a 2-cell

$$\begin{tikzcd}
& t \times t \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\beta"] & \\
t^t \times t \arrow[ur, "\ell \times t"] \arrow[rr, bend right, "ev"'] & {} & t.
\end{tikzcd}$$

This is an example of a *higher-order rewrite system* in the sense of Nipkow [14]. Nipkow's definition is formally different, but his higher-order rewrite systems are in bijection with 2-signatures $h\colon X_2 \to \mathcal{L}_1(X)_{||}$ such that for all rules $r \in X_2$, letting $(\Gamma \vdash M, N\colon A) = h(r)$:

- M is not a variable,
- A is a sort,
- each variable occurring in Γ occurs free in M.

These restrictions help formulating and proving decidability problems on higher-order rewrite systems, whose extension to our setting we leave open.

Let us now anticipate over our main results below and state our soundness and completeness theorem. Given a higher-order rewrite system X, i.e., a 2-signature satisfying the above conditions, let $\mathcal{R}(X)$ be the following locally-preordered 2-category. It has:

- objects are types in $\mathcal{L}_0(X_0)$;
- morphisms $A \to B$ are λ-terms in $\mathcal{L}_1(X)(A \vdash B)$, modulo $\beta\eta$;
- given two parallel morphisms M and N, there is one 2-cell $M \to N$ exactly when there is a sequence of reductions $M \to^* N$ in the usual sense [14].

Proposition 5. *$\mathcal{R}(X)$ is 2-cartesian closed.*

$\mathcal{R}(X)$ and $\mathcal{H}(X)$ have the same objects and morphisms. But because our inference rules for forming reductions are the same as deduction rules for proving the existence of a reduction in the usual sense, we may send any reduction $P \colon M \to N$ to the unique reduction $M \to N$ in $\mathcal{R}(X)$.

Theorem 1 (Soundness and completeness)**.** *This defines an identity-on-objects, identity-on-morphisms, locally full cartesian closed 2-functor $\mathcal{R}(X) \xrightarrow{!} \mathcal{H}(X)$.*

4.2. **Theories with binding.** Understanding reduction rules as equations, it is easy to define the free cartesian closed category generated by a 2-signature. This yields an adjunction

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{H}'"] \arrow[r, phantom, "\perp"] & \mathsf{CCCat}. \arrow[l, bend left, "\mathcal{W}'"] \end{tikzcd} \tag{4.1}$$

This adjunction provides a categorical semantics for theories with binding, which is more general than other approaches by Fiore and Hur [6], Hirschowitz and Maggesi [8], and Zsidó [18], and which is in line with Lambek's seminal paper [11].

If I understand correctly, the motivation for Fiore and Hur's subtle approach is the will to explain the λ-calculus by strictly less than itself. The present framework does not obey this specification, and instead tends to view the λ-calculus as a universal (parameterised) theory with binding.

We end this section by giving a formal construction of the adjunction (4.1). Cartesian closed categories form a full, reflective subcategory of $\mathsf{2CCCat}$, via the functor $\mathcal{J} \colon \mathsf{2CCCat} \to \mathsf{CCCat}$ sending a cartesian closed 2-category $\mathcal{C}$ to the cartesian closed category with:

- objects those of $\mathcal{C}$,
- morphisms those of $\mathcal{C}$, modulo the congruence generated by $f \sim g$ iff there exists a 2-cell $f \to g$.

Here, $\mathcal{J}(\mathcal{C})$ is thought of as the free cartesian closed 2-category with trivial 2-cells (i.e., 0 or 1). The desired adjunction is obtained by composing the adjunctions

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{H}"] \arrow[r, phantom, "\perp"] & \mathsf{2CCCat} \arrow[l, bend left, "\mathcal{W}"] \arrow[r, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathsf{CCCat}. \arrow[l, bend left] \end{tikzcd}$$

4.3. **Non-examples.** Non-examples are given by calculi whose reduction semantics is defined on terms modulo a so-called *structural congruence*, e.g., CCS [12], or the π-calculus [5, 13].

For example, consider the CCS term $(a \mid 0) \mid \overline{a}$. In CCS, it is *structurally* equivalent to $(a \mid \overline{a}) \mid 0$, which then reduces to $0 \mid 0$.

In order to account for this, we would have to consider a 2-signature with reduction rules for structural congruence, here $(M_1 \mid M_2) \mid M_3 \to M_1 \mid (M_2 \mid M_3)$ for associativity, and $M \mid N \to N \mid M$ for commutativity. But then, these reductions count as proper reductions, which departs from the desired computational behaviour. For example, the term $a \mid a$ has an infinite reduction sequence, using commutativity.

Anticipating the development in the next sections, a potential solution is to extend 2-signatures to *2-theories*. For any 2-signature X, let $X_{||}$ denote the set of pairs of reduction rules r, s with a common type $G \vdash M \to N \colon A$. A 2-theory is a 2-signature X, together with a set of equations between parallel reductions, i.e., a subset X_3 of $\mathcal{L}(X)_{||}$.

The main adjunction announced above (1.1) extends to an adjunction between 2-theories and cartesian closed 2-categories. Using equations, we may specify that any reduction $M \to M$ using only structural rules be the identity on M, and consider the computational behaviour of a 2-category to consist of its non-invertible morphisms, as proposed by Hilken [7]. A question is whether for a given calculus this can be done with finitely many equations.

5. A 2-lambda-calculus

We now begin the construction of Adjunction (1.1). We start in this section by defining a monad $\mathcal{L}$ on Sig, which we will use to factor Adjunction (1.1) as

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{L}"] \arrow[r, phantom, "\perp"] & \mathcal{L}\text{-Alg} \arrow[l, bend left, "\mathcal{U}"] \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathsf{2CCCat}, \arrow[l, bend left, "\mathcal{V}"] \end{tikzcd}$$

where:

- $\mathcal{L}$-Alg is the category of $\mathcal{L}$-algebras,
- $\mathcal{L} \colon \mathsf{Sig} \to \mathcal{L}$-Alg is a shortcut for $X \mapsto (\mathcal{L}^2 X \xrightarrow{\mu} \mathcal{L}X)$,
- $\mathcal{U}(\mathcal{L}X \xrightarrow{h} X) = X$,
- $\mathsf{2CCCat}$ is the category of cartesian closed 2-categories, which we define in Section 6.

The left-hand adjunction holds by $\mathcal{L}$ being a monad, thus we concentrate in Section 7 on establishing the right-hand one.

But for now, let us define the monad $\mathcal{L}$.

5.1. **Syntax.** Given a 2-signature $X = ((X_0, X_1), h \colon X_2 \to \mathcal{L}_1(X)_{||})$ (actually $\mathcal{L}_1(X)$ is $\mathcal{L}_1(X_0, X_1)$), we construct a new 2-signature $\mathcal{L}(X)$, whose reduction rules represent reduction sequences in the "higher-order rewrite system" defined by X, modulo permutation equivalence. The 2-signature $\mathcal{L}(X)$ has the same base 1-signature (X_0, X_1), and as reduction rules the terms of a 2λ-calculus (in the sense of Hilken [7]) modulo permutation equivalence, which we now define.

First, terms, called *reductions*, are defined by induction in Figure 1. The typing judgement has the shape $\Gamma \vdash P : M \to N \colon A$, where A is a type in $\mathcal{L}_0(X_0)$, Γ is a list of pairs of a variable and a type, with no variable appearing more than once, M and N are terms of type $\Gamma \vdash A$ modulo $\beta\eta$, and P is a reduction. In the sequel, we often forget the variables in such pairs $(\Gamma \vdash A)$, and identify them with sequents in $\mathcal{S}_0(X_0)$.

When clear from context, we abbreviate substitutions $[M_1/x_1, \ldots, M_n/x_n]$ by $[M_1, \ldots, M_n]$. For a context G, G_i denotes its ith type. Also, for $(M, N) \in \mathcal{L}_1(X)_{||}$, we let $X(M, N)$ be the set of all reduction rules $r \in X_2$ such that $h(r) = (M, N)$. We write $X(\Gamma \vdash M, N \colon A)$ to indicate the common type of M and N. Similarly, $X(G \vdash A)$ denotes the set of operations in X_1 above $G \vdash A$.

$$\frac{\dots \qquad \Gamma \vdash P_i : M_i \to N_i \colon G_i \qquad \dots}{\Gamma \vdash r(P_1, \dots, P_n) : M[M_1, \dots, M_n] \to N[N_1, \dots, N_n] \colon A} \; (r \in X(G \vdash M, N \colon A))$$

$$\frac{\Gamma \vdash P : M_1 \to M_2 \colon A \qquad \Gamma \vdash Q : M_2 \to M_3 \colon A}{\Gamma \vdash P \,;_{M_2} Q : M_1 \to M_3 \colon A} \qquad \Gamma, x \colon A, \Delta \vdash x : x \to x \colon A$$

$$\Gamma \vdash () : () \to () \colon 1$$

$$\frac{\Gamma \vdash P_1 : M_1 \to N_1 \colon G_1 \qquad \dots \qquad \Gamma \vdash P_n : M_n \to N_n \colon G_n}{\Gamma \vdash c(P_1, \dots, P_n) : c(M_1, \dots, M_n) \to c(N_1, \dots, N_n) \colon A} \; (c \in X_1(G \vdash A))$$

$$\frac{\Gamma, x \colon A \vdash P : M \to N \colon B}{\Gamma \vdash \lambda x \colon A.P : \lambda x \colon A.M \to \lambda x \colon A.N \colon B^A}$$

$$\frac{\Gamma \vdash P : M \to M' \colon B^A \qquad \Gamma \vdash Q : N \to N' \colon A}{\Gamma \vdash PQ : MN \to M'N' \colon B}$$

$$\frac{\Gamma \vdash P : M \to M' \colon A \qquad \Gamma \vdash Q : N \to N' \colon B}{\Gamma \vdash (P, Q) : (M, N) \to (M', N') \colon A \times B}$$

$$\frac{\Gamma \vdash P : M \to N \colon A \times B}{\Gamma \vdash \pi_{A,B} P : \pi_{A,B} M \to \pi_{A,B} N \colon A} \qquad \frac{\Gamma \vdash P : M \to N \colon A \times B}{\Gamma \vdash \pi'_{A,B} P : \pi'_{A,B} M \to \pi'_{A,B} N \colon B}$$

FIGURE 1. Reductions

5.2. **Substitution.** Next, we define substitution, which has "type"

$$\frac{\Gamma \vdash Q : N \to N' \colon \Delta \qquad \Delta \vdash P : M \to M' \colon A}{\Gamma \vdash P[Q] : M[N] \to M'[N'] \colon A,} \tag{5.1}$$

i.e., given a reduction P and a tuple of reductions Q, it produces a reduction of the indicated type, which we denote $P[Q]$. Here, we denote by $\Gamma \vdash Q : N \to N' \colon \Delta$ a tuple of reductions $\Gamma \vdash Q_i : N_i \to N'_i \colon \Delta_i$, for $1 \leq i \leq |\Delta|$.

The definition is a bit tricky:

- first we define *left whiskering*, which has "type"
$$\frac{\Gamma \vdash Q : N \to N' \colon \Delta \qquad \Delta \vdash M \colon A}{\Gamma \vdash M[Q] : M[N] \to M'[N'] \colon A,}$$
- then we define *right whiskering*, which has "type"
$$\frac{\Gamma \vdash N \colon \Delta \qquad \Delta \vdash P : M \to M' \colon A}{\Gamma \vdash P[N] : M[N] \to M'[N] \colon A,}$$
(where N denotes a tuple),
- then we define substitution by
$$P[Q] = (P[N] \,;_{M'[N]} M'[Q]).$$

There is of course another legitimate definition, namely

$$P[Q] = (M[Q] \,;_{M[N']} P[N']).$$

The two will be equated by permutation equivalence in the next section.

Left whiskering is defined by induction, with $\Delta = (x_1 \colon A_1, \ldots, x_n \colon A_n)$ and $Q = (Q_1, \ldots, Q_n)$, by:

$$\begin{array}{rcl} ()[Q] & = & () \\ x_i[Q] & = & Q_i \\ c(M_1, \ldots, M_p)[Q] & = & c(M_1[Q], \ldots, M_p[Q]) \\ (\lambda x \colon B.M)[Q] & = & \lambda x \colon B.(M[Q, x]) \quad (\text{for } x \notin \mathrm{dom}(\Delta)) \\ (MN)[Q] & = & (M[Q]N[Q]) \\ (M, N)[Q] & = & (M[Q], N[Q]) \\ (\pi_{A,B}M)[Q] & = & \pi_{A,B}(M[Q]) \\ (\pi'_{A,B}M)[Q] & = & \pi'_{A,B}(M[Q]). \end{array}$$

Right whiskering is defined by induction, with $\Delta = (x_1 \colon A_1, \ldots, x_n \colon A_n)$ and $N = (N_1, \ldots, N_n)$, by:

$$\begin{array}{rcl} (r(P_1, \ldots, P_p))[N] & = & r(P_1[N], \ldots, P_p[N]) \\ (P_1 ;_{M''} P_2)[N] & = & (P_1[N] ;_{M''[N]} P_2[N]) \\ ()[N] & = & () \\ x_i[N] & = & N_i \\ c(P_1, \ldots, P_p)[N] & = & c(P_1[N], \ldots, P_p[N]) \\ (\lambda x \colon B.P')[N] & = & \lambda x \colon B.(P'[N, x]) \quad (\text{for } x \notin \mathrm{dom}(\Delta)) \\ (P_1P_2)[N] & = & (P_1[N]P_2[N]) \\ (P_1, P_2)[N] & = & (P_1[N], P_2[N]) \\ (\pi_{A,B}P')[N] & = & \pi_{A,B}(P'[N]) \\ (\pi'_{A,B}P')[N] & = & \pi'_{A,B}(P'[N]). \end{array}$$

Definition 3. *Let* $P[Q] = (P[N] ;_{M'[N]} M'[Q])$.

Proposition 6. *Given reductions P and Q as above, the capture-avoiding substitution $P[Q]$ is a well-typed reduction* $\Gamma \vdash P[Q] : M[N] \to M'[N'] \colon A$.

Similarly, there is a weakening operation with "type"

$$\frac{\Gamma \vdash P : M \to N \colon A}{\Gamma, x \colon B \vdash P : M \to N \colon A.} \ (x \notin \Gamma)$$

5.3. **Permutation equivalence.** We now define *permutation equivalence* on reductions, by the equations in Figures 3 and 4, in Appendix A. The *congruence* rules in Figure 3 are bureaucratic: they just say that permutation equivalence is a congruence. The *category* rules make reductions of a given type $\Gamma \vdash A$ into a category. In Figure 4, the *beta and eta* rules mirror the term-level beta and eta rules. Finally, the *lifting* rules lift composition of reductions towards toplevel.

So, $\mathcal{L}(X)$ has sorts X_0, operations X_1, and as reduction rules in $\mathcal{L}(X)(G \vdash M, N \colon A)$ all reductions $G \vdash P : M \to N \colon A$, modulo the equations.

This easily extends to:

Proposition 7. $\mathcal{L}$ *is a functor* $\mathsf{Sig} \to \mathsf{Sig}$.

Now, consider $\mathcal{L}\mathcal{L}(X)$. We define a mapping $\mu_X \colon \mathcal{L}\mathcal{L}(X) \to \mathcal{L}(X)$, by induction on reductions. The typing rule for reduction rules specialises to:

$$\frac{\begin{array}{c}(R \in \mathcal{L}(X)(G \vdash M, N \colon A)) \\ \Gamma \vdash P_1 : M_1 \to N_1 \colon G_1 \quad \ldots \quad \Gamma \vdash P_n : M_n \to N_n \colon G_n\end{array}}{\Gamma \vdash R(P_1, \ldots, P_n) : M[M_1, \ldots, M_n] \to N[N_1, \ldots, N_n] \colon A} \ .$$

We set $\mu(R(P_1,\dots,P_n)) = R[\mu(P_1),\dots,\mu(P_n)]$. The other cases just propagate the substitution:

$$\begin{array}{rcl}
P\,;Q & \mapsto & \mu(P)\,;\mu(Q)\\
x & \mapsto & x\\
() & \mapsto & ()\\
c(P_1,\dots,P_n) & \mapsto & c(\mu(P_1),\dots,\mu(P_n))\\
\lambda x\colon A.P & \mapsto & \lambda x\colon A.\mu(P)\\
PQ & \mapsto & \mu(P)\mu(Q)\\
(P,Q) & \mapsto & (\mu(P),\mu(Q))\\
\pi P & \mapsto & \pi(\mu(P))\\
\pi' P & \mapsto & \pi'(\mu(P)).
\end{array}$$

Lemma 1. *This defines a natural transformation $\mu\colon \mathfrak{L}^2 \to \mathfrak{L}$, which makes the diagram*

$$\begin{tikzcd}
\mathfrak{L}^3 \arrow[r, "\mathfrak{L}\mu"] \arrow[d, "\mu\mathfrak{L}"'] & \mathfrak{L}^2 \arrow[d, "\mu"] \\
\mathfrak{L}^2 \arrow[r, "\mu"'] & \mathfrak{L}
\end{tikzcd}$$

commute.

Similarly, there is a natural transformation $\eta\colon id \to \mathfrak{L}$, sending each $r \in X(G \vdash M, N\colon A)$ to the reduction $G \vdash r(x_1,\dots,x_n) : M \to N\colon A$, and we have:

Lemma 2. *The diagram*

$$\begin{tikzcd}
\mathfrak{L} \arrow[r, "\eta\mathfrak{L}"] \arrow[dr, equal] & \mathfrak{L}^2 \arrow[d, "\mu"] & \mathfrak{L} \arrow[l, "\mathfrak{L}\eta"'] \arrow[dl, equal] \\
& \mathfrak{L} &
\end{tikzcd}$$

commutes.

Corollary 1. $(\mathfrak{L},\mu,\eta)$ *is a monad on* Sig.

A crucial result is:

Proposition 8. *For all $\Gamma \vdash Q : N \to N'\colon \Delta$ and $\Delta \vdash P : M \to M'\colon A$, we have:*

$$\Gamma \vdash P[Q] \equiv (M[Q]\,;_{M[N']} P[N']) : M[N] \to M'[N']\colon A.$$

Proof. We proceed by induction on P. Most cases are bureaucratic. Consider for instance $P = c(P_1,\dots,P_p)$. Then, by definition:

$$P[Q] = (c(P_1[N],\dots,P_p[N])\,;_{c(M'_1[N],\dots,M'_p[N])} c(M'_1[Q],\dots,M'_p[Q]).$$

By the third lifting rule, this is $\equiv$-related to

$$c(P_1[N]\,;_{M'_1[N]} M'_1[Q],\dots,P_p[N]\,;_{M'_p[N]} M'_p[Q]).$$

By p applications of the induction hypothesis, we obtain

$$c(M_1[Q]\,;_{M_1[N']} P_1[N'],\dots,M_p[Q]\,;_{M_p[N']} P_p[N']),$$

which by lifting again yields the desired result:

$$c(M_1[Q],\dots,M_p[Q])\,;_{c(M_1[N'],\dots,M_p[N'])} c(P_1[N'],\dots,P_p[N']).$$

The case where something actually happens is $P = r(P_1,\dots,P_p)$, with $r \in X(G \vdash M_0, M'_0\colon A)$ and each $\Delta \vdash P_i : M_i \to M'_i\colon G_i$. Then, the left-hand side is

$$r(P_1[N],\dots,P_p[N])\,;_{M_0[M_1,\dots,M_n][N]} M'_0[M'_1,\dots,M'_p][Q].$$

By lifting, omitting indices of vertical compositions, we have

$$r(P_1[N],\dots,P_p[N]) \equiv r(M_1[N],\dots,M_p[N])\ ;\ M'_0[P_1[N],\dots,P_p[N]].$$

Observing that $M'_0[M'_1,\dots,M'_p][Q] = M'_0[M'_1[Q],\dots,M'_p[Q]]$, the whole is $\equiv$-related to

$$\begin{array}{l} r(M_1[N],\dots,M_p[N]); \\ M'_0[P_1[N],\dots,P_p[N]]; \\ M'_0[M'_1[Q],\dots,M'_p[Q]], \end{array}$$

i.e., by lifting (inductively):

$$\begin{array}{l} r(M_1[N],\dots,M_p[N]); \\ M'_0[(P_1[N]\ ;\ M'_1[Q]),\dots,(P_p[N]\ ;\ M'_1[Q])]. \end{array}$$

This is by induction hypothesis $\equiv$-related to

$$\begin{array}{l} r(M_1[N],\dots,M_p[N]); \\ M'_0[(M_1[Q]\ ;\ P_1[N']),\dots,(M_1[Q]\ ;\ P_p[N'])], \end{array}$$

i.e., by lifting again to

$$\begin{array}{l} r(M_1[N],\dots,M_p[N]); \\ M'_0[M_1[Q],\dots,M_1[Q]]; \\ M'_0[P_1[N'],\dots,P_p[N']]. \end{array}$$

The second lifting rule then yields

$$\begin{array}{l} r(M_1[Q],\dots,M_p[Q]); \\ M'_0[P_1[N'],\dots,P_p[N']]. \end{array}$$

And then the first lifting rule yields

$$\begin{array}{l} M_0[M_1[Q],\dots,M_1[Q]]; \\ r(M_1[N'],\dots,M_p[N']); \\ M'_0[P_1[N'],\dots,P_p[N']], \end{array}$$

so, by the second lifting rule again:

$$\begin{array}{l} M_0[M_1[Q],\dots,M_1[Q]]; \\ r(P_1[N'],\dots,P_p[N']), \end{array}$$

i.e., the right-hand side. □

6. Cartesian closed 2-categories

6.1. **Definition.** In a 2-category $\mathcal{C}$, a diagram $A \xleftarrow{p} C \xrightarrow{q} B$ is a *product diagram* iff for all object D, the induced functor

$$\mathcal{C}(D,C) \xrightarrow{\Delta} \mathcal{C}(D,C)\times\mathcal{C}(D,C) \xrightarrow{\mathcal{C}(D,p)\times\mathcal{C}(D,q)} \mathcal{C}(D,A)\times\mathcal{C}(D,B)$$

is an isomorphism. Because this family of functors is 2-natural in D, the inverse functors will also be 2-natural.

Similarly, an object 1 of $\mathcal{C}$ is *terminal* iff for all D the unique functor

$$\mathcal{C}(D,1) \xrightarrow{!} 1$$

is an isomorphism (where the right-hand 1 is the terminal category).

Definition 4. *A* 2-category with finite products*, or fp 2-category, is a 2-category $\mathcal{C}$, equipped with a terminal object and a 2-functor*

$$\mathcal{C}\times\mathcal{C} \xrightarrow{\times} \mathcal{C},$$

plus, for all A and B, a product diagram

$$A \xleftarrow{p} A\times B \xrightarrow{q} B.$$

In such an fp 2-category $\mathcal{C}$, given objects A and B, an *exponential* for them is a pair of an object B^A and a morphism $ev\colon A \times B^A \to B$, such that for all D, the functor

$$\begin{tikzcd}
 & \mathcal{C}(A,A) \times \mathcal{C}(D,B^A) \arrow[r, "\times"] & \mathcal{C}(A \times D, A \times B^A) \arrow[dr, "\mathcal{C}(A \times D, ev)"] & \\
\mathcal{C}(D,B^A) \arrow[ur, "(id_A!, id)"] & & & \mathcal{C}(A \times D, B)
\end{tikzcd}$$

is an isomorphism. As above, because this family of functors is 2-natural in D, the inverse functors will also be 2-natural.

Definition 5. *A* cartesian closed 2-category*, or cartesian closed 2-category, is an fp 2-category, equipped with a choice of exponentials for all pairs of objects. The category* $\mathsf{2CCCat}$ *has cartesian closed 2-categories as objects, and stricly structure-preserving functors between them as morphisms.*

7. Main adjunction

7.1. **Right adjoint.** Given a cartesian closed 2-category $\mathcal{C}$, define $\mathcal{V}(\mathcal{C}) = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$ as follows. First, let as in Section 2 $(\mathcal{C}_0, \mathcal{C}_1) = \mathcal{V}_1(\mathcal{C})$, and recall the canonical $\mathcal{L}_0$ and $\mathcal{L}_1$-algebra structures h_0 and h_1. Let then the reduction rules in $\mathcal{C}_2(G \vdash M, N\colon A)$ be the 2-cells in $\mathcal{C}(h_0(G), h_0(A))(h_1(M), h_1(N))$, abbreviated to $\mathcal{C}(G, A)(M, N)$ in the sequel.

This signature $\mathcal{V}\mathcal{C}$ has a canonical $\mathcal{L}$-algebra structure $h_2\colon \mathcal{L}(\mathcal{V}\mathcal{C}) \to \mathcal{V}\mathcal{C}$, which we define by induction over terms in Figure 2. In the case for λ, φ denotes the structure isomorphism $\mathcal{C}((\prod \Gamma) \times A, B) \cong \mathcal{C}(\prod \Gamma, B^A)$.

In order for the definition to make sense as a morphism $\mathcal{L}(\mathcal{V}\mathcal{C}) \to \mathcal{V}\mathcal{C}$, we have to check its compatibility with the equations. We have first:

Lemma 3. *For all* $\Delta \vdash Q : N \to N' \colon \Gamma$ *and* $\Gamma \vdash P : M \to M' \colon A$ *in* $\mathcal{L}(\mathcal{V}\mathcal{C})$,

$$\begin{tikzcd}
\Delta \arrow[r, bend left, "{M[N]}"] \arrow[r, bend right, "{M'[N']}"'] \arrow[r, phantom, "\Downarrow h_2(P[Q])"] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\Delta \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow h_2(Q)"] & \Gamma \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow h_2(P)"] & A.
\end{tikzcd}$$

Proof. By induction on P and the axioms for cartesian closed 2-categories. $\square$

Lemma 4. *Any two equated reductions are mapped to the same 2-cell in* $\mathcal{C}$.

Proof. We proceed by induction on the proof of the considered equation. The rules of Figure 3 hold because, in $\mathcal{C}$, vertical composition is associative and unital, and equality is a congruence. The beta rule is less easy, so we spell it out.

The left-hand reduction is interpreted in $\mathcal{C}$ as

$$\begin{tikzcd}
\prod \Gamma \arrow[r, bend left, "{(\varphi M, N)}"] \arrow[r, bend right, "{(\varphi M', N')}"'] \arrow[r, phantom, "\Downarrow (\varphi P, Q)"] & B^A \times A \arrow[r, "ev"] & B
\end{tikzcd}$$

which is equal to

$(G \vdash x_i : x_i \to x_i \colon G_i) \mapsto (id_{\pi_i} \colon \pi_i \to \pi_i \colon \prod G \to G_i)$
$(G \vdash () : () \to () \colon 1) \mapsto (id_! \colon ! \to ! \colon \prod G \to 1)$
$(\Gamma \vdash c(P_1, \dots, P_n) : c(M_1, \dots, M_n) \to c(N_1, \dots, N_n) \colon A) \mapsto$

$$\begin{tikzcd}
\prod \Gamma \arrow[r, bend left, "{(M_1, \dots, M_n)}"] \arrow[r, bend right, "{(N_1, \dots, N_n)}"'] \arrow[r, phantom, "\Downarrow P"] & \prod G \arrow[r, "c"] & A
\end{tikzcd} \quad (c \in \mathcal{C}_1(G, A), P = (P_1, \dots, P_n))$$

$(\Gamma \vdash r(P_1, \dots, P_n) : M[M_1, \dots, M_n] \to N[N_1, \dots, N_n] \colon A) \mapsto$

$$\begin{tikzcd}
\prod \Gamma \arrow[r, bend left, "{(M_1, \dots, M_n)}"] \arrow[r, bend right, "{(N_1, \dots, N_n)}"'] \arrow[r, phantom, "\Downarrow P"] & \prod G \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow r"] & A
\end{tikzcd} \quad (P = (P_1, \dots, P_n))$$

$$(G \vdash P \,;_{M_2} Q : M_1 \to M_3 \colon A) \mapsto \begin{tikzcd}
\prod G \arrow[r, bend left=50, "M_1"] \arrow[r] \arrow[r, bend right=50, "M_3"'] \arrow[r, phantom, shift left=3, "\Downarrow P"] \arrow[r, phantom, shift right=3, "\Downarrow Q"] & A
\end{tikzcd}$$

$(\Gamma \vdash \lambda x \colon A.P : \lambda x \colon A.M \to \lambda x \colon A.N \colon B^A) \mapsto \varphi(P \colon M \to N \colon (\prod \Gamma) \times A \to B)$

$$(\Gamma \vdash PQ : MN \to M'N' \colon B) \mapsto \begin{tikzcd}
\prod \Gamma \arrow[r, bend left, "{(M, N)}"] \arrow[r, bend right, "{(M', N')}"'] \arrow[r, phantom, "\Downarrow (P,Q)"] & B^A \times A \arrow[r, "ev"] & B
\end{tikzcd}$$

$$(\Gamma \vdash (P, Q) : (M, N) \to (M', N') \colon A \times B) \mapsto \begin{tikzcd}
\prod \Gamma \arrow[r, bend left, "{(M, N)}"] \arrow[r, bend right, "{(M', N')}"'] \arrow[r, phantom, "\Downarrow (P,Q)"] & A \times B
\end{tikzcd}$$

$$(\Gamma \vdash \pi_{A,B} P : \pi_{A,B} M \to \pi_{A,B} N \colon A) \mapsto \begin{tikzcd}
\prod \Gamma \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow P"] & A \times B \arrow[r, "\pi"] & A
\end{tikzcd}$$

$$(\Gamma \vdash \pi'_{A,B} P : \pi'_{A,B} M \to \pi'_{A,B} N \colon B) \mapsto \begin{tikzcd}
\prod \Gamma \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow P"] & A \times B \arrow[r, "\pi'"] & B
\end{tikzcd}$$

FIGURE 2. The $\mathcal{L}$-algebra structure on $\mathcal{V}(\mathcal{C})$

$$\begin{tikzcd}[column sep=large]
\prod\Gamma \arrow[r, bend left, "{(id,N)}"] \arrow[r, bend right, "{(id,N')}"'] \arrow[r, phantom, "\Downarrow (id,Q)"] & \prod\Gamma\times A \arrow[r, bend left, "\varphi M\times A"] \arrow[r, bend right, "\varphi M'\times A"'] \arrow[r, phantom, "\Downarrow \varphi P\times A"] & B^A\times A \arrow[r, "ev"] & B
\end{tikzcd}$$

which is in turn equal (by cartesian closedness of $\mathcal{C}$) to:

$$\begin{tikzcd}[column sep=large]
\prod\Gamma \arrow[r, bend left, "{(id,N)}"] \arrow[r, bend right, "{(id,N')}"'] \arrow[r, phantom, "\Downarrow (id,Q)"] & \prod\Gamma\times A \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow P"] & B
\end{tikzcd}$$

and hence to the right-hand side of the equation by Lemma 3. The other beta and eta rules similarly hold by the properties of products, internal homs, and terminal object in $\mathcal{C}$.

The lifting rules hold by (particular cases of) the interchange law in $\mathcal{C}$ and functoriality of the structural isomorphisms

$$\mathcal{C}(A\times B, C)\cong\mathcal{C}(B, C^A) \qquad \text{and} \qquad \mathcal{C}(C, A\times B)\cong\mathcal{C}(C,A)\times\mathcal{C}(C,B),$$

which concludes the proof. □

This assignment extends to cartesian closed functors and we have:

Proposition 9. *$\mathcal{V}$ is a functor* $\mathsf{2CCCat}\to\mathsf{Sig}$.

7.2. **Left adjoint.** Given an $\mathcal{L}$-algebra $h\colon\mathcal{L}(X)\to X$, we now construct a cartesian closed 2-category $\mathcal{F}(X,h)$. It has:

- objects the types in $\mathcal{L}_0(X_0)$;
- 1-cells $A\to B$ the terms in $\mathcal{L}_1(X_0,X_1)(A,B)$;
- 2-cells $M\to N\colon A\to B$ the reduction rules in $X_2(M,N)$.

We then must define the cartesian closed 2-category structure, and we start with the 2-category structure. Composition of 1-cells $A\xrightarrow{M}B\xrightarrow{N}C$ is defined to be $A\xrightarrow{N[M]}C$. Vertical composition of 2-cells

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left=40, "M_1"] \arrow[r, "M_2" description] \arrow[r, bend right=40, "M_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B
\end{tikzcd}$$

is given by $h(\eta(\alpha)\,;_{M_2}\,\eta(\beta))$.

Horizontal composition of 2-cells

$$\begin{tikzcd}[column sep=large]
A \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow\beta"] & C
\end{tikzcd} \tag{7.1}$$

is obtained as $h(\beta(\eta(\alpha)))$.

To show that this yields a 2-category structure, the only non obvious point is the interchange law. We deal with it using the following series of results. First, consider the *left whiskering*

$$\begin{tikzcd} A \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "N"] & C \end{tikzcd}$$

of a 2-cell α by a 1-cell N, i.e., the composition $id_N \circ \alpha = h((h(N))(\eta(\alpha)))$.

Lemma 5. *We have:* $h((h(N))(\eta(\alpha))) = h(N[\eta(\alpha)])$.

Proof. Indeed, consider the term $N(\eta(\eta(\alpha)))$ in $\mathfrak{L}(\mathfrak{L}(X))$. Its images by $h \circ \mathfrak{L}(h)$ and $h \circ \mu$ coincide, and are respectively $h((h(N))(\eta(\alpha)))$, i.e., $id_N \circ \alpha$, and $h(N[\eta(\alpha)])$. □

Similarly, consider the *right whiskering*

$$\begin{tikzcd} A \arrow[r, "M"] & B \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd}$$

of a 2-cell γ by a 1-cell M, i.e., the composition $\gamma \circ id_N = h(\gamma(\eta(h(M))))$.

Lemma 6. *We have:* $h(\gamma(\eta(h(M)))) = h(\gamma(M))$.

Proof. Consider $(\eta\gamma)(\eta M)$ in $\mathfrak{L}(\mathfrak{L}(X))$. Its images by $h \circ \mathfrak{L}(h)$ and $h \circ \mu$ coincide, and are respectively $h(\gamma(\eta(h(M))))$ and $h(\gamma(M))$. □

Now, we prove that the two sensible ways of mimicking horizontal composition using whiskering coincide with actual horizontal composition:

Lemma 7. *For any cells as in* (7.1),

$$(\beta \circ id_M) ; (id_{N'} \circ \alpha) = \beta \circ \alpha = (id_N \circ \alpha) ; (\beta \circ id_{M'}).$$

Proof. Consider first the reduction $\eta(\beta(M)) ; \eta(N'[\eta(\alpha)])$ in $\mathfrak{L}(\mathfrak{L}(X))$. Taking $h \circ \mathfrak{L}(h)$ and $h \circ \mu$ as above respectively yields

- $h(\eta(h(\beta(M))) ; \eta(h(N'[\alpha])))$, and
- $h(\beta(M) ; N'[\eta(\alpha)]) = h(\beta(\eta(\alpha)))$,

hence the left-hand equality. Then consider $\eta(N[\eta(\alpha)]) ; \eta(\gamma(M'))$. Evaluating as before yields the right-hand equality. □

Finally, consider any configuration like:

$$\begin{tikzcd} A \arrow[r, bend left=40, "M"] \arrow[r, "M'" description] \arrow[r, bend right=40, "M''"'] & B \arrow[r, "N"] & C. \end{tikzcd}$$

with $\Downarrow\alpha$ between M and M', and $\Downarrow\beta$ between M' and M''.

Lemma 8. *We have* $(id_N \circ \alpha) ; (id_N \circ \beta) = id_N \circ (\alpha ; \beta)$.

Proof. Consider $\eta(N[\eta(\alpha)]) ; \eta(N[\eta(\beta)])$. Evaluating yields equality of

- $h(\eta(h(N[\eta(\alpha)])) ; \eta(h(N[\eta(\beta)])))$, i.e., the left-hand side, and
- $h(N[\eta(\alpha)] ; N[\eta(\beta)])$, i.e., $h(N[\eta(\alpha) ; \eta(\beta)])$ by lifting.

But now consider $N[\eta(\eta(\alpha) ; \eta(\beta))]$. Evaluating yields equality of

- $h(N[\eta(\alpha) ; \eta(\beta)])$, as above, and
- $h(N[\eta(h(\eta(\alpha) ; \eta(\beta)))])$, i.e., $h(N[\eta(\alpha ; \beta)])$ (where $\alpha ; \beta$ denotes vertical composition in our candidate 2-category), i.e., the right-hand side. □

Lemma 9. *The interchange law holds, i.e., for all reduction rules as in*

$$\begin{tikzcd}[column sep=large]
A \arrow[r, bend left=40, "M_1"] \arrow[r, "M_2" near start] \arrow[r, bend right=40, "M_3"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & B \arrow[r, bend left=40, "N_1"] \arrow[r, "N_2" near start] \arrow[r, bend right=40, "N_3"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\theta"] & C,
\end{tikzcd}$$

we have

$$(\gamma \,;\, \theta) \circ (\alpha \,;\, \beta) = (\gamma \circ \alpha) \,;\, (\theta \circ \beta).$$

Proof. By the previous results, we have

$$\begin{array}{l}
(\gamma \,;\, \theta) \circ (\alpha \,;\, \beta) \\
= ((\gamma \,;\, \theta) \circ M_1) \,;\, (N_3 \circ (\alpha \,;\, \beta)) \\
= (\gamma \circ M_1) \,;\, (\theta \circ M_1) \,;\, (N_3 \circ \alpha) \,;\, (N_3 \circ \beta) \\
= (\gamma \circ M_1) \,;\, (N_2 \circ \alpha) \,;\, (\theta \circ M_2) \,;\, (N_3 \circ \beta) \\
= (\gamma \circ \alpha) \,;\, (\theta \circ \beta).
\end{array}$$

$\square$

Now, let us show cartesian closedness. We have a bijection of hom-sets $\mathcal{L}_1(X)(C \vdash A \times B) \cong \mathcal{L}_1(X)(C \vdash A) \times \mathcal{L}_1(X)(C \vdash B)$, given by

$$\begin{array}{rcl}
\mathcal{L}_1(X)(C \vdash A \times B) & \to & \mathcal{L}_1(X)(C \vdash A) \times \mathcal{L}_1(X)(C \vdash B) \\
M & \mapsto & \pi M, \pi' M
\end{array}$$

and

$$\begin{array}{rcl}
\mathcal{L}_1(X)(C \vdash A) \times \mathcal{L}_1(X)(C \vdash B) & \to & \mathcal{L}_1(X)(C \vdash A \times B) \\
M, N & \mapsto & (M, N).
\end{array}$$

These are mutually inverse thanks to the beta and eta rules for products in the simply-typed λ-calculus.

On 2-hom-sets, we have

$$\begin{array}{rcl}
\mathcal{L}(X)(C \vdash M, N \colon A \times B) & \to & \mathcal{L}(X)(C \vdash \pi M, \pi N \colon A) \times \mathcal{L}(X)(C \vdash \pi' M, \pi' N \colon B) \\
P & \mapsto & \pi P, \pi' P
\end{array}$$

and (omitting C)

$$\begin{array}{rcl}
\mathcal{L}(X)(M_1, N_1 \colon A) \times \mathcal{L}(X)(M_2, N_2 \colon B) & \to & \mathcal{L}(X)((M_1, M_2), (N_1, N_2) \colon A \times B) \\
P_1, P_2 & \mapsto & (P_1, P_2),
\end{array}$$

which are mutually inverse thanks to the beta and eta rules for products in Figure 4. We use these to define the desired isomorphism (u, v)

$$X_2(C \vdash M, N \colon A \times B) \cong X_2(C \vdash \pi M, \pi N \colon A) \times X_2(C \vdash \pi' M, \pi' N \colon B),$$

as in the diagrams

$$\begin{tikzcd}
X_2(M, N) \arrow[r, "u"] \arrow[d, "\eta"'] & X_2(\pi M, \pi N) \times X_2(\pi' M, \pi' N) \\
\mathcal{L}(X)(M, N) \arrow[r, "\cong"'] & \mathcal{L}(X)(\pi M, \pi N) \times \mathcal{L}(X)(\pi' M, \pi' N) \arrow[u, "h \times h"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_2(\pi M, \pi N) \times X_2(\pi' M, \pi' N) \arrow[r, "v"] \arrow[d, "\eta \times \eta"'] & X_2(M, N) \\
\mathcal{L}(X)(\pi M, \pi N) \times \mathcal{L}(X)(\pi' M, \pi' N) \arrow[r, "\cong"'] & \mathcal{L}(X)(M, N). \arrow[u, "h"']
\end{tikzcd}$$

Starting from $r \in X_2(M, N)$, we obtain

$$v(u(r)) = h(\eta(h(\pi(\eta(r)))), \eta(h(\pi'(\eta(r))))).$$

But consider $(\eta(\pi\eta(r)), \eta(\pi'\eta(r)))$ in $\mathcal{L}(\mathcal{L}X)$; its images by $h \circ \mathcal{L}h$ and $h \circ \mu$ are respectively:

- $h(\eta(h(\pi(\eta r))), \eta(h(\pi'(\eta r))))$, and
- $h(\pi\eta(r), \pi'\eta(r))$, i.e., $h(\eta(r))$, i.e., r,

which must be equal because h is an $\mathcal{L}$-algebra, hence $v \circ u = id$.

Conversely, starting from $(r, s) \in X_2(M_1, M_2) \times X_2(N_1, N_2)$, we obtain the pair with components

$$h(\pi(\eta(h(\eta(r), \eta(s))))) \qquad \text{and} \qquad h(\pi'(\eta(h(\eta(r), \eta(s))))).$$

Considering $\pi(\eta(\eta(r), \eta(s))) \in \mathcal{L}(\mathcal{L}(X))$, its images by $h \circ \mathcal{L}(h)$ and $h \circ \mu$ are respectively:

- $h(\pi(\eta(h(\eta(r), \eta(s)))))$, and
- $h(\pi(\eta(r), \eta(s))) = h(\eta(r)) = r$.

As above, they must be equal, and by symmetry the second component is s, and we have proved $u \circ v = id$. Similar reasoning for the terminal object and internal homs leads to:

Proposition 10. *This yields a cartesian closed 2-category structure on $\mathcal{C}$.*

This extends to morphisms of $\mathcal{L}$-algebras, so we have constructed a functor $\mathcal{F}\colon \mathcal{L}\text{-Alg} \to \mathsf{2CCCat}$.

7.3. **Adjunction.** Consider any $\mathcal{L}$-algebra (X, h). What does $(Y, k) = \mathcal{V}(\mathcal{F}(X, h))$ look like? Sorts in Y_0 are types in $\mathcal{L}_0(X_0)$. Operations $Y_1(G \vdash A)$ are terms in $\mathcal{L}_1(X_0, X_1)(\prod G \vdash A)$. Reduction rules in $Y_2(G \vdash M, N\colon B)$ are reductions in $\mathcal{L}(X)(\prod G \vdash M', N'\colon B)$, where $M' = M[\pi_1 x/x_1, \ldots, \pi_n x/x_n]$ (and similarly for N').

Let η_X send:

- each sort $\iota \in X_0$ to the type $\iota \in \mathcal{L}_0(X_0)$,
- each operation $c \in X(G \vdash A)$ to the term $c(\pi_1 x, \ldots, \pi_n x)$, and
- each reduction rule $r \in X_2(G \vdash M, N\colon A)$ to the reduction $x\colon \prod G \vdash r(\pi_1 x, \ldots, \pi_n x) : M' \to N'\colon A$.

Theorem 2. *This η is a natural transformation which is the unit of an adjunction*

$$\begin{tikzcd} \mathcal{L}\text{-Alg} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathsf{2CCCat}. \arrow[l, bend left, "\mathcal{V}"] \end{tikzcd}$$

Proof. Consider any morphism $f\colon (X, h) \to \mathcal{V}(\mathcal{C})$, and let $(Y, k) = \mathcal{V}(\mathcal{F}(X, h))$ and $\mathcal{V}(\mathcal{C}) = (\mathcal{C}_0, \mathcal{C}_1, h_2\colon \mathcal{C}_2 \to \mathcal{C}_1)$. We now define a uniquely determined cartesian closed functor $f'\colon \mathcal{F}(X, h) \to \mathcal{C}$ making the triangle

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & \mathcal{V}(\mathcal{F}(X)) \arrow[d, "\mathcal{V}(f')"] \\ & \mathcal{V}(\mathcal{C}) \end{tikzcd}$$

commute.

On objects, it is determined by induction: on sorts by f_0, and on type constructors by the requirement that f' be cartesian closed. On morphisms, it is similarly determined by f_1 and f' being cartesian closed. On 2-cells, define f' to be $f_2 \colon X_2(A \vdash M, N \colon B) \to \mathcal{C}(f'(A), f'(B))(f'(M), f'(N))$, which is also the only possible choice from f.

We thus only have to show that f' is cartesian closed, which follows by f being a morphism of $\mathcal{L}$-algebras. For example, to show that binary products of reductions are preserved, consider $r \in X_2(C \vdash M_1, M_2 \colon A)$ and $s \in X_2(C \vdash N_1, N_2 \colon B)$. Their product in $\mathcal{F}(X)$ is obtained by considering the atomic reductions $x \colon C \vdash r(x) : M_1 \to M_2 \colon A$ and $x \colon C \vdash s(x) : N_1 \to N_2 \colon B$ and taking $h(r(x), s(x))$, which is sent by f_2 to $f_2(h(r(x), s(x)))$. But, because f is a morphism of $\mathcal{L}$-algebras, this is the same as $h_2((f_2(r))(x), (f_2(s))(y))$, which is by definition (i.e., Figure 2) the product $(f_2(r), f_2(s))$ in $\mathcal{C}$. □

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Appendix A. Equations on reductions

Current address: CNRS, Université de Savoie

Congruence

$$\frac{\Gamma \vdash P : M \to N \colon A}{\Gamma \vdash P \equiv P : M \to N \colon A} \qquad \frac{\Gamma \vdash P \equiv Q : M \to N \colon A}{\Gamma \vdash Q \equiv P : M \to N \colon A}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 : M \to N \colon A \qquad \Gamma \vdash P_2 \equiv P_3 : M \to N \colon A}{\Gamma \vdash P_1 \equiv P_3 : M \to N \colon A}$$

$$\frac{\Gamma \vdash P \equiv P' : M_1 \to M_2 \colon A \qquad \Gamma \vdash Q \equiv Q' : M_2 \to M_3 \colon A}{\Gamma \vdash (P \,;_{M_2} Q) \equiv (P' \,;_{M_2} Q') : M_1 \to M_3 \colon A}$$

$$\frac{\begin{array}{c}(r \in X(G \vdash M, N \colon A)) \\ \Gamma \vdash P_1 \equiv Q_1 : M_1 \to N_1 \colon G_1 \qquad \ldots \qquad \Gamma \vdash P_n \equiv Q_n : M_n \to N_n \colon G_n\end{array}}{\Gamma \vdash r(P_1, \ldots, P_n) \equiv r(Q_1, \ldots, Q_n) : M[M_1, \ldots, M_n] \to N[N_1, \ldots, N_n] \colon A}$$

$$\frac{\begin{array}{c}(c \in X(G \vdash A)) \\ \Gamma \vdash P_1 \equiv Q_1 : M_1 \to N_1 \colon G_1 \qquad \ldots \qquad \Gamma \vdash P_n \equiv Q_n : M_n \to N_n \colon G_n\end{array}}{\Gamma \vdash c(P_1, \ldots, P_n) \equiv c(Q_1, \ldots, Q_n) : c(M_1, \ldots, M_n) \to c(N_1, \ldots, N_n) \colon A}$$

$$\frac{\Gamma, x \colon A \vdash P \equiv Q : M \to N \colon B}{\Gamma \vdash (\lambda x \colon A.P) \equiv (\lambda x \colon A.Q) : \lambda x \colon A.M \to \lambda x \colon A.N \colon B^A}$$

$$\frac{\Gamma \vdash P \equiv P' : M \to M' \colon B^A \qquad \Gamma \vdash Q \equiv Q' : N \to N' \colon A}{\Gamma \vdash (PQ) \equiv (P'Q') : MN \to M'N' \colon B}$$

$$\frac{\Gamma \vdash P \equiv P' : M \to M' \colon A \qquad \Gamma \vdash Q \equiv Q' : N \to N' \colon B}{\Gamma \vdash (P, Q) \equiv (P', Q') : (M, N) \to (M', N') \colon A \times B}$$

$$\frac{\Gamma \vdash P \equiv Q : M \to N \colon A \times B}{\Gamma \vdash (\pi_{A,B} P) \equiv (\pi_{A,B} Q) : \pi_{A,B} M \to \pi_{A,B} N \colon A}$$

$$\frac{\Gamma \vdash P \equiv Q : M \to N \colon A \times B}{\Gamma \vdash (\pi'_{A,B} P) \equiv (\pi'_{A,B} Q) : \pi'_{A,B} M \to \pi'_{A,B} N \colon A}$$

Category

$$\frac{\Gamma \vdash P_1 : M_1 \to M_2 \colon A \qquad \Gamma \vdash P_2 : M_2 \to M_3 \colon A \qquad \Gamma \vdash P_3 : M_3 \to M_4 \colon A}{\Gamma \vdash (P_1 \,;_{M_2} (P_2 \,;_{M_3} P_3)) \equiv ((P_1 \,;_{M_2} P_2) \,;_{M_3} P_3) : M_1 \to M_4 \colon A}$$

$$\frac{\Gamma \vdash P : M \to N \colon A}{\Gamma \vdash (P \,;_N N) \equiv P : M \to N \colon A} \qquad \frac{\Gamma \vdash P : M \to N \colon A}{\Gamma \vdash (M \,;_M P) \equiv P : M \to N \colon A}$$

FIGURE 3. Equations on reductions (Congruence and category)

Beta and eta

$$\frac{\Gamma, x\colon A \vdash P : M \to M'\colon B \qquad \Gamma \vdash Q : N \to N'\colon A}{\Gamma \vdash ((\lambda x\colon A.P)Q) \equiv P[Q/x] : (\lambda x\colon A.M)N \to M'[N'/x]\colon B}$$

$$\frac{\Gamma \vdash P \equiv M \to N\colon B^A}{\Gamma \vdash P \equiv \lambda x\colon A.(Px) : M \to N\colon B^A}\ (x \notin \Gamma)$$

$$\frac{\Gamma \vdash P : M_1 \to M_2\colon A \qquad \Gamma \vdash Q : N_1 \to N_2\colon B}{\Gamma \vdash \pi(P, Q) \equiv P : \pi(M_1, N_1) \to M_2\colon A}$$

$$\frac{\Gamma \vdash P : M_1 \to M_2\colon A \qquad \Gamma \vdash Q : N_1 \to N_2\colon B}{\Gamma \vdash \pi'(P, Q) \equiv P : \pi'(M_1, N_1) \to N_2\colon A}$$

$$\frac{\Gamma \vdash P : (M_1, N_1) \to (M_2, N_2)\colon A \times B}{\Gamma \vdash P \equiv (\pi P, \pi' P) : (M_1, N_1) \to (M_2, N_2)\colon A \times B} \qquad \frac{\Gamma \vdash P : M \to N\colon 1}{\Gamma \vdash P \equiv () : M \to N\colon 1}$$

Lifting

$$\frac{(r \in X(\Gamma \vdash \langle M_1, M_2\rangle\colon A)) \qquad \Delta \vdash P : N_1 \to N_2\colon \Gamma \qquad \Delta \vdash Q : N_2 \to N_3\colon \Gamma}{\Gamma \vdash r(P \,;_{N_2} Q) \equiv M_1[P] \,;_{M_1[N_2]} r(Q) : M_1[N_1] \to M_2[N_3]\colon A}$$

$$\frac{(r \in X(\Gamma \vdash \langle M_1, M_2\rangle\colon A)) \qquad \Delta \vdash P : N_1 \to N_2\colon \Gamma \qquad \Delta \vdash Q : N_2 \to N_3\colon \Gamma}{\Gamma \vdash r(P \,;_{N_2} Q) \equiv r(P) \,;_{M_2[N_2]} M_2[Q] : M_1[N_1] \to M_2[N_3]\colon A}$$

$$\frac{\Gamma \vdash P : M_1 \to M_2\colon G \qquad \Gamma \vdash Q : M_2 \to M_3\colon G}{\Gamma \vdash (c(P \,;_{M_2} Q)) \equiv (c(P) \,;_{c(M_2)} c(Q)) : M_1 \to M_3\colon A}\ (c \in X(G \vdash A))$$

$$\frac{\Gamma, x\colon A \vdash P : M_1 \to M_2\colon B \qquad \Gamma, x\colon A \vdash Q : M_2 \to M_3\colon B}{\begin{array}{r}\Gamma \vdash (\lambda x\colon A.(P \,;_{M_2} Q)) \equiv ((\lambda x\colon A.P) \,;_{\lambda x\colon A.M_2} (\lambda x\colon A.Q))\\ : \lambda x\colon A.M_1 \to \lambda x\colon A.M_3\colon B^A\end{array}}$$

$$\frac{\begin{array}{c}\Gamma \vdash P : M_1 \to M_2\colon B^A\\ \Gamma \vdash P' : M_2 \to M_3\colon B^A \qquad \Gamma \vdash Q : N_1 \to N_2\colon A \qquad \Gamma \vdash Q' : N_2 \to N_3\colon A\end{array}}{\Gamma \vdash ((P \,;_{M_2} P')(Q \,;_{N_2} Q')) \equiv ((PQ) \,;_{M_2 N_2} (P'Q')) : M_1 N_1 \to M_3 N_3\colon B}$$

$$\frac{\begin{array}{c}\Gamma \vdash P : M_1 \to M_2\colon A\\ \Gamma \vdash P' : M_2 \to M_3\colon A \qquad \Gamma \vdash Q : N_1 \to N_2\colon B \qquad \Gamma \vdash Q' : N_2 \to N_3\colon B\end{array}}{\begin{array}{r}\Gamma \vdash ((P \,;_{M_2} P'), (Q \,;_{N_2} Q')) \equiv ((P, Q) \,;_{(M_2, N_2)} (P', Q'))\\ : (M_1, N_1) \to (M_3, N_3)\colon A \times B\end{array}}$$

$$\frac{\Gamma \vdash P : M_1 \to M_2\colon A \times B \qquad \Gamma \vdash Q : M_2 \to M_3\colon A \times B}{\Gamma \vdash (\pi_{A,B}(P \,;_{M_2} Q)) \equiv (\pi_{A,B} P \,;_{\pi_{A,B} M_2} \pi_{A,B} Q) : M_1 \to M_3\colon A}$$

$$\frac{\Gamma \vdash P : M_1 \to M_2\colon A \times B \qquad \Gamma \vdash Q : M_2 \to M_3\colon A \times B}{\Gamma \vdash (\pi'_{A,B}(P \,;_{M_2} Q)) \equiv (\pi'_{A,B} P \,;_{\pi'_{A,B} M_2} \pi'_{A,B} Q) : M_1 \to M_3\colon B}$$

FIGURE 4. Equations on reductions (beta-eta and lifting)